

Dynamics of Induced Systems

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December, 2014

Abstract

In this paper, we study the dynamical properties of actions on the space of compact subsets of the phase space. More precisely, if X is a metric space, let 2^X denote the space of non-empty compact subsets of X provided with the Hausdorff topology. If f is a continuous self-map on X , there is a naturally induced continuous self-map f_* on 2^X . Our main theme is the interrelation between the dynamics of f and f_* . For such a study, it is useful to consider the space $\mathcal{C}(K, X)$ of continuous maps from a Cantor set K to X provided with the topology of uniform

convergence, and f_* induced on $\mathcal{C}(K, X)$ by composition of maps. We mainly study the properties of transitive points of the induced system $(2^X, f_*)$ both topologically and dynamically, and give some examples. We also look into some more properties of the system $(2^X, f_*)$.

1 Introduction

We first review some useful results and set up notation. All our basic definitions and notations are as in Akin [1], though we make full attempt to explain each of these as we proceed. All our spaces are assumed to be nonempty.

Recall that any two perfect, compact, zero-dimensional metric spaces are homeomorphic and we will refer to any such as a Cantor set. A Polish space is a separable space which admits a complete metric. As a compact metric space is complete and separable, it is Polish. A Polish space with no isolated points is called perfect. Any perfect Polish space contains a Cantor set. A G_δ subset of a Polish space is Polish. It follows that any nonempty open subset of a perfect Polish space contains a Cantor set. For an exposition of this material, see, e.g. Akin [2].

Every compact metric space is a continuous image of a Cantor set. We include the brief proof.

Proposition 1.1 *If X is a nonempty, compact, metrizable space and K is a Cantor set then there is a continuous map from K onto X .*

Proof: Let $\mathcal{B} = \{U_1, U_2, \dots\}$ be a countable base for X and let Q be the closure in $X \times \{0, 1\}^\mathbb{N}$ of $\{(x, a) : a_i = 1 \Leftrightarrow x \in U_i\}$. Clearly, Q is contained in the closed set $\{(x, a) : a_i = 1 \text{ and } x \in \bar{U}_i \text{ or } a_i = 0 \text{ and } x \in X \setminus U_i\}$. It follows that if $(x_1, a_1), (x_2, a_2) \in Q$ and $U_i \in \mathcal{B}$ with $x_1 \in U_i, x_2 \in X \setminus U_i$ then $(a_1)_i = 1, (a_2)_i = 0$. Hence, the projection map from Q to $\{0, 1\}^\mathbb{N}$ is injective and so Q is a zero-dimensional compact metric space. Clearly, $\pi_1 : Q \rightarrow X$ is surjective. $Q \times K$ is perfect and so there is a homeomorphism $h : K \rightarrow Q \times K$. $\pi_1 \circ h : K \rightarrow X$ is surjective.

□

For a metric space X , we follow Illanes and Nadler [9] letting 2^X denote the space of nonempty compact subsets of X . Notice that we are excluding $\emptyset \subset X$ which is sometimes regarded as an isolated point of 2^X . If K is a compact metrizable space we let $\mathcal{C}(K, X)$ denote the space of continuous

maps from K to X . Each of these has a natural metric induced from d on X . (We will use d for all the metrics which occur, allowing context to make the referent clear.)

On 2^X we use the Hausdorff metric: For $A, B \in 2^X$

$$d(A, B) = \max\{d(a, B) : a \in A\} \cup \{d(b, A) : b \in B\}, \quad (1.1)$$

where $d(a, B) = \min\{d(a, b) : b \in B\}$. Thus, $d(A, B) < \epsilon$ if and only if each set is in the open ϵ neighborhood of the other, or, equivalently, each point of A is within ϵ of a point in B and vice-versa.

When X is compact, we occasionally use an equivalent topology on 2^X . Define for any collection $\{U_i : 1 \leq i \leq n\}$ of opene (= open and nonempty) sets,

$$< U_1, U_2, \dots, U_n > = \{E \in 2^X : E \subseteq \bigcup_{i=1}^n U_i, E \cap U_i \neq \emptyset, 1 \leq i \leq n\} \quad (1.2)$$

The topology on 2^X , generated by such collection as basis, is known as the *Vietoris topology*.

As topological objects, 2^X have a very rich structure. It is known that if X contains a *Peano continuum* then 2^X contains the Hilbert Cube. We refer the reader to Illanes and Nadler [9] or Schori and West [14] for more details.

On $\mathcal{C}(K, X)$ we use the sup metric: For $u, v \in \mathcal{C}(K, X)$

$$d(u, v) = \max\{d(u(x), v(x)) : x \in K\}, \quad (1.3)$$

with the topology of uniform convergence.

If $\{A_n\}$ is a sequence of closed sets in a topological space then

$$\begin{aligned} \overline{\bigcup_n \{A_n\}} &= \bigcup_n \{A_n\} \cup \text{Lim sup}_n \{A_n\}, \\ \text{where } \text{Lim sup}_n \{A_n\} &= \bigcap_k \overline{\bigcup_{n \geq k} \{A_n\}}. \end{aligned} \quad (1.4)$$

The results stated below are standard. One can refer to [1, 9, 12] for more details. For the sake of completeness we briefly discuss the proofs of the results which we shall use and in the process establish some of our notations.

Lemma 1.2 For a metric space X , and $\{A_n\}$ a sequence in 2^X ,

- (a) If $\{A_n\}$ converges to A in 2^X then $\overline{\bigcup_n \{A_n\}}$ is compact.
- (b) If $\overline{\bigcup_n \{A_n\}}$ is compact and $\{A_n\}$ is Cauchy then A_n converges to $\text{Limsup}\{A_n\}$.
- (c) If X is complete then 2^X is complete.
- (d) If X is compact then 2^X is compact.

Proof: We consider the sequence $\{A_n\}$ in 2^X for a metric space X .

(a) Let $\{x_n\}$ be a sequence in $\overline{\bigcup_n \{A_n\}}$. If there is an $N \in \mathbb{N}$ such that infinitely many x_i are in the compact set $\bigcup_{n \leq N} A_n$ then clearly $\{x_n\}$ has a convergent subsequence. Otherwise, since $A_n \rightarrow A$, it follows from the definition of the Hausdorff metric that there are y_i in A with $d(x_i, y_i) \rightarrow 0$. Since A is compact, $\{y_i\}$ has a convergent subsequence, and the corresponding subsequence of $\{x_i\}$ converges to the same point.

(b) Let $\epsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence there is an $N > 0$ such that all the A_n for $n \geq N$ are within $\epsilon/3$ neighbourhood of each other, and so within $2\epsilon/3$ of $\overline{\bigcup_n \{A_n\}}$. Now $\bigcup_{k \geq N} A_k \rightarrow \text{Limsup} A_n$ as $k \rightarrow \infty$, so A_n is within ϵ neighbourhood of $\text{Limsup} A_n$ when $n \geq N$. It follows that $\lim A_n = \text{Limsup} A_n$. In particular if $A_n \rightarrow A$ then $A = \text{Limsup} A_n$.

(c) Let $\{A_n\}$ be a Cauchy sequence in 2^X . We show that $\overline{\bigcup_n A_n}$ is compact. Since a compact metric space is complete, this will imply that $\{A_n\}$ converges. Now since $\overline{\bigcup_n A_n}$ is closed, it is sufficient to show that it is totally bounded. Let $\epsilon > 0$. Again there is an $N \in \mathbb{N}$ such that all A_n for $n \geq N$ are within $\epsilon/3$ neighbourhood of A_N . Therefore a finite subset of A_N which is $\epsilon/3$ dense in A_N is ϵ dense in $\overline{\bigcup_{n \geq N} A_n}$. Since $\bigcup_{n < N} A_n$ is compact, there is a finite ϵ dense set in $\overline{\bigcup_n A_n}$.

(d) If X is compact, and $\epsilon > 0$ then there exists D a finite subset of X which is ϵ -dense, i.e. for every $x \in X$ there exists $d \in D$ such that $d(x, d) < \epsilon$. If $A \in 2^X$ then the set of points of D which are less than ϵ from a point of A form an element of 2^X which has distance less than ϵ from A . Thus, the dense set of finite subsets of 2^X are also ϵ -dense. Thus 2^X is totally bounded as well as complete, and so is compact.

□

We will let (n) denote $\{1, \dots, n\}$ and write $X^{(n)}$ for the n -fold product of copies of X . Hence, $f^{(n)}$ denotes the n -fold product of copies of f as opposed to f^n which is the n -fold iterate of a map f on X .

Proposition 1.3 *Let X be a metric space and K be a compact metrizable space.*

- (i) *Define $i_X : X \rightarrow 2^X$ by $x \mapsto \{x\}$ and $i_X : X \rightarrow \mathcal{C}(K, X)$ by $x \mapsto c_x$ where c_x is the function which takes the constant value x . Both maps are isometric inclusions.*
- (ii) *Define $i_{X,n} : X^{(n)} \rightarrow 2^X$ by $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$ and $\vee : 2^{2^X} \rightarrow 2^X$ by $F \mapsto \bigcup F$. Each of these has Lipschitz constant 1, where on the product space $X^{(n)}$ we use the max of the distances between corresponding coordinates. Hence, the map $(2^X)^{(n)} \rightarrow 2^X$ given by $(A_1, \dots, A_n) \mapsto A_1 \cup \dots \cup A_n$ has Lipschitz constant 1. The image of $i_{X,n}$ is closed and \vee is surjective.*
- (iii) *For $D \subset X$ let $FIN(D) \subset 2^X$ be the collection of all finite subsets of D . For \mathcal{A} a clopen partition of K we let $\mathcal{C}(\mathcal{A}, D) \subset \mathcal{C}(K, X)$ denote the functions with values in D which are constant on each element of \mathcal{A} . If D is dense in X then $FIN(D)$ is dense in 2^X . If K is a Cantor set and $\{\mathcal{A}_n\}$ is a sequence of clopen partitions whose mesh tends to zero then $\bigcup_n \{\mathcal{C}(\mathcal{A}_n, D)\}$ is dense in $\mathcal{C}(K, X)$. In particular, if X is separable then so are $\mathcal{C}(K, X)$ and 2^X . Furthermore, if X has no isolated points then neither does 2^X and if K is a Cantor set then $\mathcal{C}(K, X)$ has no isolated points either.*
- (iv) *The map $Im : \mathcal{C}(K, X) \rightarrow 2^X$ defined by $Im(u) = u(K)$ has Lipschitz constant 1. If K is a Cantor set then Im is surjective.*
- (v) *If X is connected, then 2^X is connected and if K is a Cantor set then $\mathcal{C}(K, X)$ is connected. If X is a Cantor set then 2^X is a Cantor set and $\mathcal{C}(K, X)$, too, is totally disconnected.*
- (vi) *For $h : K_1 \rightarrow K_2$ a continuous map of compact metrizable spaces, the map $h^* : \mathcal{C}(K_2, X) \rightarrow \mathcal{C}(K_1, X)$ defined by $h^*(u) = u \circ h$ has Lipschitz constant at most 1. If h is surjective then h^* is an isometric inclusion and $Im(h^*(u)) = Im(u)$.*
- (vii) *For $f : X_1 \rightarrow X_2$ a continuous map of metric spaces, the maps $f_* : \mathcal{C}(K, X_1) \rightarrow \mathcal{C}(K, X_2)$ and $f_* : 2^{X_1} \rightarrow 2^{X_2}$ are defined by $u \mapsto f \circ u$ and $A \mapsto f(A)$, respectively. If f is uniformly continuous or continuous, then the maps f_* are uniformly continuous or continuous, respectively and $f_*(Im(u)) = Im(f_*(u))$.*

(viii) The following subsets are closed.

$$\begin{aligned}
INT &= \{(A, B) \in 2^X \times 2^X : A \cap B \neq \emptyset\}, \\
INC &= \{(A, B) \in 2^X \times 2^X : A \subset B\}, \\
EPS &= \{(x, A) \in X \times 2^X : x \in A\},
\end{aligned} \tag{1.5}$$

If Y is a closed subset of X then 2^Y is closed, regarded as a subset of 2^X . If U is an open subset of X then 2^U is open, regarded as a subset of 2^X , and $2^X \setminus 2^{X \setminus U} = \{A : A \cap U \neq \emptyset\}$ is open.

Proof: The isometry and Lipschitz constant results in (i), (ii), (iv) and (vi) are easy to check. If $F \in 2^{2^X}$ and $\{x_n\}$ is a sequence in $\bigcup F$, then there exists $A_n \in F$ such that $x_n \in A_n$. Since F is compact we can assume by going to a subsequence that $\{A_n\}$ converges to $A \in F$. By Lemma 1.3 $\bigcup F \supset \bigcup_n \{A_n\} \cup A = \overline{\bigcup_n \{A_n\}}$ is compact and so $\{x_n\}$ has a subsequence which converges to a point of $\bigcup F$. Hence, the latter is compact. When K is a Cantor set, Im is surjective by Proposition 1.1. Since $\vee \circ i_{2^X} = 1_{2^X}$, \vee is surjective. If A contains $n+1$ distinct points at least 2ϵ apart then A has distance at least ϵ from every set of cardinality at most n , i.e. from the image of $i_{X,n}$. Hence, the latter image is closed.

The density results in (iii) are clear. It follows that if D is a countable dense set in X then $FIN(D)$ is a countable dense set in 2^X and $\bigcup_n \{\mathcal{C}(\mathcal{A}_n, D)\}$ is a countable dense subset of $\mathcal{C}(K, X)$ when K is a Cantor set and $\{\mathcal{A}_n\}$ is a sequence of clopen partitions with mesh tending to zero. To approximate $u \in \mathcal{C}(K, X)$ we apply uniform continuity of u with respect to any metric on K . If K is not a Cantor set we apply Proposition 1.1 to get $h : K_1 \rightarrow K$ be a continuous surjection with K_1 a Cantor set. Then h^* is an isometric inclusion of $\mathcal{C}(K, X)$ onto a subset of the separable space $\mathcal{C}(K_1, X)$. Any subset of a separable metric space is separable. If X has no isolated points then neither does $FIN(X)$ or $\mathcal{C}(\mathcal{A}, X)$ for any clopen partition \mathcal{A} . If X is connected then each $\mathcal{C}(\mathcal{A}, X)$ is homeomorphic to $X^{(n)}$ where n is the cardinality of \mathcal{A} and so is connected. By choosing a sequence $\{\mathcal{A}_n\}$ of clopen partitions where each refines its predecessor and with the mesh tending to zero we obtain a dense subset of $\mathcal{C}(K, X)$ as an increasing union of connected subsets. It follows that $\mathcal{C}(K, X)$ is connected and so 2^X its image under Im is connected as well.

If X is a Cantor set then we can choose as metric on X an *ultra-metric*

which satisfies the strengthening of the triangle inequality:

$$d(x, y) \leq \max(d(x, z), d(z, x)).$$

This is equivalent to saying that $V_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$ is an open equivalence relation for all $\epsilon > 0$. This implies that the open balls are clopen. The ultrametric condition then holds for the induced metrics on 2^X and $\mathcal{C}(K, X)$ and so each of these is totally disconnected. Since 2^X is compact and perfect it is a Cantor set, completing the proof of (v).

In (vii) the results are easy when f is uniformly continuous. In general, if $\{A_n\}$ is a sequence in 2^{X_1} converging to A then Lemma 1.2(a) implies that $A \subset \overline{\bigcup_n \{A_n\}}$ and the latter is compact. So we can use uniform continuity of the restriction of f to this set to show that $\{f_*(A_n)\}$ converges to $f_*(A)$. Similarly, if $\{u_n\}$ converges to u in $\mathcal{C}(K, X_1)$ then $\{Im(u_n)\}$ converges to $Im(u)$ in 2^{X_1} and so $u(K) \subset \overline{\bigcup_n \{u_n(K)\}}$ and the latter is compact. Again apply uniform continuity of f on the subset to show that $\{f_*(u_n)\}$ converges to $f_*(u)$.

For (viii) it is easiest to proceed by checking that $A \cap B = \emptyset$, $A \cap (X \setminus B) \neq \emptyset$ and $x \notin A$ are open conditions.

□

For us a dynamical system (X, f) consists of a continuous map f on a metric space X , i.e. $f : X \rightarrow X$. We are essentially interested in the dynamics of the system $(2^X, f_*)$.

Such a study was first undertaken by Bauer and Sigmund [4]. The dynamics on 2^X is richer than the dynamics on X , which can be easily assessed from the results and examples in Bauer and Sigmund [4], Glasner and Weiss [7], John Banks [3] and Sharma and Nagar [16, 17]. This provides us a strong motivation for the study of dynamics of $(2^X, f_*)$.

In this article, we look into some known results - improving them, some new concepts arising thereof, and answer some natural questions with a view to further develop this study.

2 Spaces of Continuous Maps, Spaces of Compact Subsets and Induced Dynamics

For a system (X, f) we call $(2^X, f_*)$ and $(\mathcal{C}(K, X), f_*)$ the *induced systems*. The map $Im : \mathcal{C}(K, X) \rightarrow 2^X$ defines an action map between the induced

systems. When K is a Cantor set, it is a factor map.

Corollary 2.1 *Let K be Cantor set. If in a dynamical system (X, f) the periodic points are dense then the periodic points are dense in the induced systems $(2^X, f_*)$ and $(\mathcal{C}(K, X), f_*)$.*

Proof: Let PER be the dense set of periodic points in X . Any finite subset subset of PER is a periodic point of 2^X and if $u \in C(\mathcal{A}, PER)$ with \mathcal{A} a clopen partition of X , the u is a periodic point of $\mathcal{C}(K, X)$. By (iv) above, $FIN(PER)$ is dense in 2^X and $\bigcup_{\mathcal{A}} \{C(\mathcal{A}, PER)\}$ is dense in $\mathcal{C}(K, X)$.

□

Remark: Each periodic orbit is a fixed point in 2^X . So if X is infinite and the periodic points are dense, or more generally, if there are infinitely many periodic points in X then there are infinitely many fixed points in 2^X . It follows that if X is compact and contains infinitely many periodic points then the set of fixed points for the induced system on 2^X has an accumulation point. Such a system cannot, for example, be expansive.

If $A, B \subset X$ we denote by $N(A, B)$ the *hitting time set* $\{n \in \mathbb{N} : A \cap f^{-n}(B) \neq \emptyset\}$. Clearly, if f_1 and f_2 are continuous functions on X_1 and X_2 respectively then for $A_1, B_1 \subset X_1, A_2, B_2 \subset X_2$ with respect to the product map $f_1 \times f_2$ on $X_1 \times X_2$ the hitting time set $N(A_1 \times A_2, B_1 \times B_2)$ is exactly $N(A_1, B_1) \cap N(A_2, B_2)$. The system is called *topologically transitive* when $N(U, V) \neq \emptyset$ for every pair of opene sets $U, V \subset X$. While f need not be surjective, its image is clearly dense and so U opene implies $f^{-n}(U)$ is opene for every $n \in \mathbb{N}$.

The system is called *weak mixing* when the product system $(X \times X, f \times f)$ is topologically transitive. It then follows that the n -fold product system $(X^{(n)}, f^{(n)})$ is weak mixing for $n = 1, 2, \dots$. This follows by induction using the beautiful *Furstenberg Intersection Lemma* [5]. We prove a strengthening due to Karl Petersen [13].

Theorem 2.2 *For a dynamical system (X, f) , assume that $N(U, V) \cap N(U, U) \neq \emptyset$ for every pair of opene sets $U, V \subset X$ and so the system is topologically transitive. For all opene sets $U_1, V_1, U_2, V_2 \subset X$ there exist opene sets $U_3, V_3 \subset X$ such that*

$$N(U_3, V_3) \subset N(U_1, V_1) \cap N(U_2, V_2). \quad (2.1)$$

In particular, the system is weak mixing.

Proof: $N(U_1, V_1) \neq \emptyset$ implies there exists $n_1 \in \mathbb{N}$ such that $U_0 = U_1 \cap f^{-n_1}(V_1)$ is opene. $N(U_0, U_2) \neq \emptyset$ implies there exists $n_2 \in \mathbb{N}$ such that $U = U_1 \cap f^{-n_1}(V_1) \cap f^{-n_2}(U_2)$ is opene. Since f is transitive, $f^{-n_1-n_2}(V_2)$ is opene.

$$\begin{aligned}
& N(U, U) \cap N(U, f^{-n_1-n_2}(V_2)) \subset \\
& N(U_1, f^{-n_2}(U_2)) \cap N(f^{-n_1}(V_1), f^{-n_1}(f^{-n_2}(V_2))) \\
& = N(U_1, f^{-n_2}(U_2)) \cap N(f^{n_1}(f^{-n_1}(V_1)), f^{-n_2}(V_2)) \\
& \subset N(U_1, f^{-n_2}(U_2)) \cap N(V_1, f^{-n_2}(V_2)).
\end{aligned} \tag{2.2}$$

Fix $n_0 \in N(U_1, f^{-n_2}(U_2)) \cap N(V_1, f^{-n_2}(V_2))$. With $n = n_0 + n_2$ the sets $U_3 = U_1 \cap f^{-n}(U_2)$, $V_3 = V_1 \cap f^{-n}(V_2)$ are opene.

Let $k \in N(U_3, V_3)$. Then $f^{-k}(V_3) \cap U_3 \neq \emptyset$. That is $f^{-k}(V_1) \cap f^{-n-k}(V_2) \cap U_1 \cap f^{-n}(U_2) \neq \emptyset$. Hence $k \in N(U_1, V_1) \cap N(f^{-n}(U_2), f^{-n}(V_2)) = N(U_1, V_1) \cap N(U_2, V_2)$.

As before, $N(f^{-n}(U_2), f^{-n}(V_2)) \subset N(U_2, V_2)$ and so (2.1) follows.

□

As a consequence, we obtain the following.

Lemma 2.3 *If f_i is a continuous map on a metric space X_i for $i = 1, \dots, k$ and the product system $(2^{X_1} \times \dots \times 2^{X_k}, (f_1)_* \times \dots \times (f_k)_*)$ is transitive then the product system $(X_1 \times \dots \times X_k, f_1 \times \dots \times f_k)$ is weak mixing.*

Proof: : Given opene $U, V \subset X_1 \times \dots \times X_n$ we show that $N(U, U) \cap N(U, V) \neq \emptyset$ and then apply Theorem 2.2.

Choose opene $U_i, V_i \subset X_i$ such that $U_1 \times \dots \times U_k \subset U$ and $V_1 \times \dots \times V_k \subset V$. Let $\bar{U}_i = 2^{U_i}$ and $\bar{V}_i = \{A \in 2^{X_i} : A \cap U_i \neq \emptyset\} \cap \{A \in 2^{X_i} : A \cap V_i \neq \emptyset\}$. Since $(f_1)_* \times \dots \times (f_k)_*$ is transitive there exist $A_i \in \bar{U}_i$ and $n \in \mathbb{N}$ such that $((f_i)_*)^n(A_i) \in \bar{V}_i$ for $i = 1, \dots, k$. So there exist $x_i, y_i \in A_i$ with $(f_i)^n(x_i) \in U_i, f^n(y_i) \in V_i$. Since $A_i \in 2^{U_i}$, $x_i, y_i \in U_i$. Hence, $n \in N(U_1 \times \dots \times U_k, U_1 \times \dots \times U_k) \cap N(U_1 \times \dots \times U_k, V_1 \times \dots \times V_k)$ and so $n \in N(U, U) \cap N(U, V)$.

□

An action map $\pi : (X_1, f_1) \rightarrow (X_2, f_2)$ is a continuous map $\pi : X_1 \rightarrow X_2$ such that $f_2 \circ \pi = \pi \circ f_1$. When π is surjective we call it a *factor map* and say that (X_2, f_2) is a factor of (X_1, f_1) . It is easy to see that a factor of a topologically transitive system is topologically transitive and so a factor of a weak mixing system is weak mixing.

We now strengthen the result in [3, 17].

Theorem 2.4 *Let K be a Cantor set. If f is a continuous map on a metric space X then the following are equivalent:*

- (a) (X, f) is weak mixing.
- (b) $(\mathcal{C}(K, X), f_*)$ is topologically transitive.
- (c) $(2^X, f_*)$ is topologically transitive.

When these conditions hold, then the product system on $(X \times \mathcal{C}(K, X) \times 2^X, f \times f_ \times f_*)$ is weak mixing and so each of the induced systems is weak mixing.*

Proof: (a) \Rightarrow (b): We prove that when (X, f) is weak mixing, then $(X \times \mathcal{C}(K, X), f \times f_*)$ is topologically transitive. Notice that if \mathcal{A} is a clopen partition of K then $\mathcal{C}(\mathcal{A}, X)$ is an invariant set and the subsystem $(\mathcal{C}(\mathcal{A}, X), f_*)$ is clearly isomorphic to the product system $(X^{(n)}, f^{(n)})$ with n the cardinality of \mathcal{A} . Hence, $(X \times \mathcal{C}(\mathcal{A}, X), f \times f_*)$ is topologically transitive for any clopen partition \mathcal{A} . Now let $U_1, V_1 \subset X, U_2, V_2 \subset \mathcal{C}(K, X)$ be opene. By choosing the mesh of \mathcal{A} fine enough we obtain a clopen partition such that $U_2 \cap \mathcal{C}(\mathcal{A}, X), V_2 \cap \mathcal{C}(\mathcal{A}, X)$ are opene relative to $\mathcal{C}(\mathcal{A}, X)$. Hence, $N(U_1 \times (U_2 \cap \mathcal{C}(\mathcal{A}, X)), V_1 \times (V_2 \cap \mathcal{C}(\mathcal{A}, X))) \subset N(U_1 \times U_2, V_1 \times V_2)$ is nonempty as required. It then follows that the factor $(\mathcal{C}(K, X), f_*)$ is topologically transitive, proving (b).

(b) \Rightarrow (c): Because K is a Cantor set, Im is a factor map and so transitivity on 2^X follows from transitivity on $\mathcal{C}(K, X)$.

(c) \Rightarrow (a): This is Lemma 2.3 with $k = 1$.

Above we showed that (X, f) weak mixing implies $(X \times \mathcal{C}(K, X), f \times f_*)$ is topologically transitive. Since the product system $(X \times X, f \times f)$ is weak mixing, we see that the product system on $X \times X \times \mathcal{C}(K, X \times X)$ is transitive. As $(\mathcal{C}(K, X \times X), (f \times f)_*)$ is isomorphic to $(\mathcal{C}(K, X) \times \mathcal{C}(K, X), f_* \times f_*)$ it follows that the system on $X \times \mathcal{C}(K, X)$ is weak mixing and so the system on $X \times \mathcal{C}(K, X) \times \mathcal{C}(K, X)$ is weak mixing. Applying Im on the third factor we see that $(X \times \mathcal{C}(K, X) \times 2^X, f \times f_* \times f_*)$ is a factor of the latter and so is weak mixing as well.

□

Corollary 2.5 *If (X, f) is weak mixing then the restriction of the product system to each of the closed, invariant subsets $INT, INC \subset 2^X \times 2^X, EPS \subset X \times 2^X$ are weak mixing systems.*

Proof: Let K be a Cantor set. We will show that each of these systems is a factor of the weak mixing system $(\mathcal{C}(K, X), f_*)$.

Let $\{K_0, K_1, K_2\}$ be a partition of K by three clopen sets. The map $\mathcal{C}(K, X) \rightarrow INT$ by $u \mapsto (u(K_0 \cup K_1), u(K_0 \cup K_2))$ is clearly an action map. If $(A, B) \in INT$ we can choose continuous surjections $u_0 : K_0 \rightarrow A \cap B$, $u_1 : K_1 \rightarrow A$, $u_2 : K_2 \rightarrow B$ and concatenate to obtain $u \in \mathcal{C}(K, X)$ mapping to (A, B) . The map $\mathcal{C}(K, X) \rightarrow INC$ by $u \mapsto (u(K_0), u(K))$ is an action map. If $(A, B) \in INC$ then choose surjections $u_0 : K_0 \rightarrow A$, $u_{12} : K_1 \cup K_2 \rightarrow B$ and concatenate.

Finally, fix $e \in K$ and map $\mathcal{C}(K, X)$ to EPS by $u \mapsto (u(e), u(K))$. Again this is an action map. If $(x, A) \in EPS$ then we can choose a continuous surjection $u_0 : K \rightarrow A$. Let $y \in K$ with $u_0(y) = x$. Since K is homogeneous (e.g. there are Cantor sets which are topological groups) we can choose a homeomorphism h on K such that $h(e) = y$. Then $u = u_0 \circ h$ maps to (x, A) .

□

3 Transitive Points for Induced Systems

For the system (X, f) and $x \in X$, the *omega limit set of x* , denoted $\omega f(x)$ is the set of limit points of the orbit sequence. Thus, $\omega f(x) = \text{Limsup}\{\{f^n(x)\} : n \in \mathbb{N}\}$. We call x a *transitive point* when $\omega f(x) = X$. If \mathcal{B} is a base for the topology then the set of transitive points, $Trans_f$ is $\bigcap \{\bigcup_{k \geq n} f^{-k}(U) : k \in \mathbb{N}, U \in \mathcal{B}\}$. Clearly, if X contains a transitive point then the system is topologically transitive. In fact, every $N(U, V)$ is infinite. Furthermore, the space X is separable and, unless X consists of a single periodic orbit, the space has no isolated points. Conversely, if X is complete as well as separable then topological transitivity implies that $Trans_f$ is a dense G_δ set. We apply the Baire category theorem to the above description of $Trans_f$ with \mathcal{B} a countable base.

Two points $x_1, x_2 \in X$ are called *asymptotic* when

$$\lim_{n \rightarrow \infty} d(f^n(x_1), f^n(x_2)) = 0.$$

In that case, $\omega f(x_1) = \omega f(x_2)$. In particular, if $x_1 \in Trans_f$ then $x_2 \in Trans_f$.

From now on we will assume that X is a complete, separable, metric space with no isolated points and so is a perfect Polish space. By Lemma

1.2 and Proposition 1.3 the same is true of 2^X and $\mathcal{C}(K, X)$. Because X is perfect, a point $x \in X$ is a transitive point if and only if the orbit sequence $\{f^n(x) : n \in \mathbb{N}\}$ is dense in X .

A single periodic orbit with more than one point is not weak mixing. If (X, f) is weak mixing then by Theorem 2.4 the induced systems are weak mixing as well and so admit transitive points. The goal of this section is to examine the characteristics of these transitive points.

Definition 3.1 *Let (X, f) be a dynamical system. A Kronecker subset L is a Cantor set contained in X such that $\{f^n|_L : n \in \mathbb{N}\}$ is dense in $C(L, X)$.*

This view of Kronecker subsets comes from Katznelson [11].

Proposition 3.2 *If L is a Kronecker set for (X, f) and L_0 is a nonempty clopen subset of L then L_0 is a Kronecker set for (X, f) .*

Proof: If u_0 is an arbitrary element of $C(L_0, X)$ and $\epsilon > 0$ then extend u_0 arbitrarily on the clopen set $L \setminus L_0$ to obtain $u \in C(L, X)$. There exists $n \in \mathbb{N}$ such that $d(u(x), f^n(x)) < \epsilon$ for all $x \in L$ and so, a fortiori, $d(u_0(x), f^n(x)) < \epsilon$ for all $x \in L_0$.

□

Theorem 3.3 *L is a Kronecker set for a system (X, f) if and only if the inclusion map $u_L : L \rightarrow X$ is a transitive point for the induced system $(\mathcal{C}(L, X), f_*)$. Equivalently, u is a transitive point for $(\mathcal{C}(K, X), f_*)$ with K a Cantor set, if and only if u is injective with $\text{Im}(u) = u(K)$ a Kronecker set for (X, f) .*

Proof: u is a transitive point for $(\mathcal{C}(K, X), f_*)$ if and only if $\{f^n \circ u : n \in \mathbb{N}\}$ is dense in $\mathcal{C}(K, X)$. Since $\mathcal{C}(K, X)$ distinguishes points of K , it follows that u must be an injective map in that case and so, by compactness, it is a homeomorphism onto its image $L = u(K)$. So $u^* : \mathcal{C}(L, X) \rightarrow \mathcal{C}(K, X)$ is an isomorphism between the induced systems. So u is a transitive point of $\mathcal{C}(K, X)$ if and only if u_L is a transitive point of $\mathcal{C}(L, X)$. The f_* orbit of u_L is $\{f^n|_L : n \in \mathbb{N}\}$ and so u_L is a transitive point if and only if L is a Kronecker subset.

□

Corollary 3.4 *If (X, f) is a dynamical system with X a complete, separable, perfect metric space, then (X, f) admits a Kronecker set if and only if the system is weak mixing.*

Proof: By Theorem 3.3 (X, f) admits a Kronecker subset if and only if the induced system on $\mathcal{C}(K, X)$ is topologically transitive. By Theorem 2.4 this occurs if and only if (X, f) is weak mixing.

□

In extending these results to the n -fold product, we recall the natural set isomorphisms $(A^B)^C \cong A^{B \times C} \cong (A^C)^B$. We thus obtain natural isometric isomorphisms:

$$\mathcal{C}(K, X^{(n)}) \cong \mathcal{C}(K \times \{1, \dots, n\}, X) \cong \mathcal{C}(K, X)^{(n)}. \quad (3.1)$$

Note that if K is a Cantor set then $K \times \{1, \dots, n\}$ is a Cantor set with clopen partition $\{K \times \{i\} : i = 1, \dots, n\}$. We will write (u_1, \dots, u_n) for an element of any of these.

Corollary 3.5 *Let $L \subset X^{(n)}$ with coordinate factors $L_i = \pi_i(L)$.*

L is a Kronecker set for $(X^{(n)}, f^{(n)})$ if and only if the following three conditions holds

- *The restriction of the coordinate projection $\pi_i : L \rightarrow L_i$ is injective and so is a homomorphism for $i = 1, \dots, n$.*
- *The subsets $\{L_1, \dots, L_n\}$ of X are pairwise disjoint.*
- *The union $L_1 \cup \dots \cup L_n$ is a Kronecker set for (X, f) .*

Conversely, if $\{L_1, \dots, L_n\}$ is a partition of a Kronecker set for (X, f) by nonempty clopen subsets then for any choice of homeomorphisms $u_i : K \rightarrow L_i$ of a fixed Cantor set K , the image $(u_1, \dots, u_n)(K) \subset X^{(n)}$ is a Kronecker set for $(X^{(n)}, f^{(n)})$.

Proof: L is a Kronecker set for $(X^{(n)}, f^{(n)})$ if and only if it is the image of some transitive point $(u_1, \dots, u_n) \in \mathcal{C}(K, X^{(n)})$ for some Cantor set K . Via the isomorphisms of (3.1) this is true if and only if (u_1, \dots, u_n) is a transitive point of $\mathcal{C}(K \times \{1, \dots, n\}, X)$ which has image $L_1 \cup \dots \cup L_n$ and is partitioned by $\{L_1, \dots, L_n\}$. By Theorem 3.3 a transitive point for either induced system is an injective map and so is a homeomorphism onto its image.

□

Now we consider $(2^X, f_*)$.

Theorem 3.6 *Let C be a transitive point for $(2^X, f_*)$ with X a complete, separable, perfect metric space.*

- (a) *If g is a continuous function which commutes with f and $C \cap g(C) \neq \emptyset$ then g is the identity map. In fact, if there exist $c_1, c_2 \in C$ with c_2 asymptotic to $g(c_1)$ then g is the identity map. The sets of the bi-infinite sequence $\{f^n(C) : n \in \mathbb{Z}\}$ are pairwise disjoint. In particular, $\{C, f(C), f^2(C), \dots\}$ is a pairwise disjoint sequence of transitive points for $(2^X, f_*)$.*
- (b) *If μ is a invariant probability measure for f , then $\mu(C) = 0$.*
- (c) *C is nowhere dense.*
- (d) *C has infinitely many components. In particular, C is infinite.*
- (e) *Every point of C is a transitive point for f .*
- (f) *C is a uniformly proximal set. If $x \in X$ then there is a sequence of iterates $f^{j_i}(C)$ which converges to $\{x\}$.*

Proof: (a) Suppose $c_1, c_2 \in C$ with $g(c_1)$ asymptotic to c_2 . If $x \in X$ then there exists a sequence j_n such that $f_*^{j_n}(C)$ converges to $\{x\}$. Then $\{f^{j_n}(c_1)\}$ converges to x as does $\{f^{j_n}(g(c_1)) = g(f^{j_n}(c_1))\}$ because it is asymptotic to $\{f^{j_n}(c_2)\}$. Hence, $x = g(x)$. Since x was arbitrary, $g = 1_X$.

If two elements of the sequence $\{f^n(C) : n \in \mathbb{Z}\}$ intersect then there exists $x \in C$ and a positive integer k such that $f^k(x) \in C$. Hence, $f^k = 1_X$. This is impossible if X is infinite and f is transitive. Finally, all of the points of the f_* orbit of the transitive point C are transitive points.

(b) Since the measure is invariant the sequence of disjoint sets $\{f^{-k}(C)\}$ all have measure $\mu(C)$. Since the sum is finite $\mu(C) = 0$.

(c) We note that $x \in X$ is a *nonwandering point* if for every neighborhood U of x , $f^n(U) \cap U \neq \emptyset$ for $n \geq 1$. The set of all nonwandering points is called the *nonwandering set* of f . Any opene set is nonwandering and so the result follows from (a). That is, if $U, V \subset C$ are disjoint opene subsets then there exists a positive integer in $N(U, V) \subset N(C, C)$ and this would contradict (a).

(d) Let $\{x_1, \dots, x_n\}$ be a set of n distinct points and $2\epsilon < \min d(x_i, x_k)$ for $i \neq k = 1, \dots, n$. There exists $j > 0$ such that $f^j(C)$ is ϵ close to $\{x_1, \dots, x_n\}$. Hence, $\{f^{-j}(V_\epsilon(x_i)) : i = 1, \dots, n\}$ is a partition of C into n open sets and hence the number of components is at least n .

(e) For every $U \subset X$ open, there exists $j > 0$ such that $(f_*)^j(C) \in C(U)$, i.e. $f^j(C) \subset U$. Hence,

$$j \in \bigcap_{x \in C} N(\{x\}, U). \quad (3.2)$$

(f) The point $\{x\}$ is a limit point of the f_* orbit of C .

□

Lemma 3.7 Assume $C \in 2^X$.

(a) If $A \in 2^X$ such that

$$\lim_{j \rightarrow \infty} \max\{d(f^j(a), f^j(C)) : a \in A\} = 0, \quad (3.3)$$

then $C \cup A$ is asymptotic to C with respect to f_* . In particular, if C is a transitive point for f_* then so is $C \cup A$.

(b) Let F be a finite subset of X with $F \cap C = \emptyset$. If $B \in 2^X$ is perfect and $(f_*)^{j_i}(C \cup F) \rightarrow B$ then $(f_*)^{j_i}(C) \rightarrow B$. If $C \cup F$ is a transitive point for f_* then so is C .

Proof: (a) $\max\{d(f^j(a), f^j(C)) : a \in A\}$ is the Hausdorff distance from $(f_*)^j(C \cup A)$ to $(f_*)^j(C)$.

(b) It suffices to show that every convergent subsequence of $f_*^{j_i}(C)$ has limit B . Assume that subsequence of $f_*^{j_i}(C)$ converges to B_1 . By going to a further subsequence we can assume that $f_*^{j_i}(F)$ converges to F_1 . Since the finite sets with cardinality at most that of F form a closed subset of 2^X , F_1 is finite. Continuity of the map \cup implies that $B_1 \cup F_1 = B$. Since B is perfect and B_1 is closed, $B_1 = B$. If $C \cup F$ is a transitive point and B is a Cantor set transitive point then B is in $\omega f_*(C \cup F)$ and so is in $\omega f_*(C)$. Hence, the latter contains $\omega f_*(B) = 2^X$.

□

Theorem 3.8 Let (C_1, \dots, C_n) be a transitive point for $(f_*)^{(n)}$ on $(2^X)^{(n)}$.

(a) The C_i 's are pairwise disjoint and so form a partition of $C = \bigcup_i C_i$. If $c_i \in C_i$ for $i = 1, \dots, n$ then (c_1, \dots, c_n) is a transitive point for $f^{(n)}$.

(b) The union $C = \bigcup_i C_i$ is a transitive point for f_* and it contains c_1, \dots, c_n such that (c_1, \dots, c_n) is a transitive point for $f^{(n)}$. The set of transitive points C for which such a decomposition exists is dense in the set of transitive points for f_* .

Proof: (a) Let $\{x_1, \dots, x_n\}$ be distinct points with $2\epsilon > \min d(x_i, x_k)$ for $i \neq k = 1, \dots, n$. There exists j such that $(f_*)^j(C_i)$ is ϵ close to $\{x_i\}$ for $i = 1, \dots, n$ and so $\{f^j(C_i) : i = 1, \dots, n\}$ are pairwise disjoint and hence $\{C_i : i = 1, \dots, n\}$ are pairwise disjoint. If $c_i \in C_i$ then $f^j(c_i)$ is ϵ close to x_i for $i = 1, \dots, n$. This proves that (c_1, \dots, c_n) is a transitive point of $f^{(n)}$.

(b) The map $(2^X)^{(n)} \rightarrow 2^X$ by $(A_1, \dots, A_n) \mapsto \bigcup_i A_i$ is surjective and continuous and so is a factor map from $(f_*)^{(n)}$ to f_* . A surjective continuous map takes dense sets to dense sets and a factor map takes transitive points to transitive points.

□

Call C an n -decomposable transitive point when there exists (C_1, \dots, C_n) a transitive point for $(f_*)^{(n)}$ on $(2^X)^{(n)}$ such that $C = \bigcup_i C_i$.

Corollary 3.9 *If C is an n -decomposable transitive point then it contains at least n distinct accumulation points.*

Proof: If C is n -decomposable then it admits a clopen partition $\{C_1, \dots, C_n\}$ with each C_i infinite and so with each containing an accumulation point.

□

We will need the following routine result:

Lemma 3.10 *Let L be a totally disconnected compact metric space. If $\{C_1, \dots, C_n\}$ are pairwise disjoint, closed nonempty subsets of L then there exists a clopen partition $\{L_1, \dots, L_n\}$ of L such that $C_i \subset L_i$ for $i = 1, \dots, n$.*

Proof: Inductively we can for $i = 1, \dots, n-1$ choose L_i a clopen subset of L which contains C_i and which is disjoint from L_j for $j < i$ and from C_j for $j > i$. Then let $L_n = L \setminus (L_1 \cup \dots \cup L_{n-1})$. Thus, $\{L_1, \dots, L_n\}$ is the required clopen partition of L .

□

A subset E of X is called *independent* if any finite sequence (e_1, e_2, \dots, e_n) of points in E is a transitive point in the product system $(X^{(n)}, f^{(n)})$. Independent sets were studied by Iwanik [10], and he proved the existence of large independent sets for weakly mixing systems. A subset of X is called *scrambled* when every pair of points in it is proximal and no pair of distinct points is asymptotic. Independent sets are also wandering sets in X , and form a scrambled set in X .

Theorem 3.11 *Let (X, f) be weak mixing and L be a Kronecker subset of X .*

- (a) *L is a transitive point for f_* which is n -decomposable for every n . In fact, if $\{L_1, \dots, L_n\}$ is a partition of L by nonempty clopen subsets then (L_1, \dots, L_n) is a transitive point for $(f_*)^{(n)}$. If c_1, \dots, c_n are distinct points in L then (c_1, \dots, c_n) is a transitive point for $f^{(n)}$, i.e. L is an independent set.*
- (b) *If C is any closed infinite subset of L of X then C is a transitive point for f_* . If $\{C_1, \dots, C_n\}$ are pairwise disjoint, closed infinite subsets of L then (C_1, \dots, C_n) is a transitive point for $(f_*)^{(n)}$.*
- (c) *If C is a closed subset of L which contains n accumulation points then it is an n -decomposable transitive point. If it contains exactly n accumulation points then it is not $(n+1)$ -decomposable.*

Proof: (a) L is a Kronecker subset if and only if the inclusion map is a transitive point for f_* on $\mathcal{C}(L, X)$. Since the map $Im : \mathcal{C}(L, X) \rightarrow 2^X$ is a factor map, it takes transitive points to transitive points. Applied to the inclusion map of L itself this implies that $L \in 2^X$ is a transitive point.

If $\{L_1, \dots, L_n\}$ is a partition of L by nonempty clopen subsets then by Proposition 3.2 each is a Kronecker set. Let $(B_1, \dots, B_n) \in (2^X)^{(n)}$. Choose $g_i \in \mathcal{C}(L_i, X)$ with $g_i(L_i) = B_i$. Concatenate to define $g \in \mathcal{C}(L, X)$. There exists $j > 0$ such that $f^j|_L$ is close to g and so $f^j(L_i)$ is close to B_i for $i = 1, \dots, n$. Hence, (L_1, \dots, L_n) is a transitive point for $(f_*)^{(n)}$. Thus, L is n -decomposable. Alternatively, one can apply Corollary 3.5.

If c_1, \dots, c_n are distinct points of L then there exists a partition L_1, \dots, L_n of L with $c_i \in L_i$ for $i = 1, \dots, n$. Since (L_1, \dots, L_n) is a transitive point for $(f_*)^{(n)}$, (c_1, \dots, c_n) is a transitive point for $f^{(n)}$ by Theorem 3.8 (a).

(b) If c_1, \dots, c_n are distinct points of L_0 then they are distinct points of L and so by (a) (c_1, \dots, c_n) is a transitive point of $f^{(n)}$.

If $\{C_1, \dots, C_n\}$ are pairwise disjoint, closed infinite subsets of L , then by Lemma 3.10 there is a clopen partition $\{L_1, \dots, L_n\}$ with $C_i \subset L_i$ for $i = 1, \dots, n$. Let $(B_1, \dots, B_n) \in (2^X)^{(n)}$ and $\epsilon > 0$. There exists $M > 0$ and points $x_{ij} \in B_i, i = 1, \dots, n, j = 1, \dots, M$ so that $\{x_{ij} : j = 1, \dots, M\}$ is $\epsilon/2$ dense in B_i for $i = 1, \dots, n$. Now choose distinct points $c_{ij} \in C_i, i = 1, \dots, n, j = 1, \dots, M$. This is where we need that each C_i is infinite. By Lemma 3.10 again there exist clopen partitions $\{L_{ij} : j = 1, \dots, M\}$ of L_i for $i = 1, \dots, n$ such that $c_{ij} \in L_{ij}$ for all i, j . Define $g \in \mathcal{C}(L, X)$ by $g(x) = x_{ij}$ for $x \in L_{ij}$. There exists $k > 0$ such that $f^k|L$ is $\epsilon/2$ close to g and so $f^k(C_i)$ is $\epsilon/2$ close to $\{x_{i1}, \dots, x_{iM}\}$ and so is ϵ close to B_i for $i = 1, \dots, n$. This shows that (C_1, \dots, C_n) is a transitive point for $(f_*)^{(n)}$.

(c) Let c_1, \dots, c_n be distinct accumulation points of C . By Lemma 3.10 there is a clopen partition $\{C_1, \dots, C_n\}$ of C with $c_i \in C_i$. Hence, each C_i is infinite. By (b) (C_1, \dots, C_n) is a $(f_*)^{(n)}$ transitive point and so C is n -decomposable. If there are only n accumulation points then C is not $(n+1)$ -decomposable by Corollary 3.9.

□

By combining Corollary 2.5 with Theorem 3.3 we sharpen the former.

Corollary 3.12 *If L is a Kronecker subset of X and $\{L_0, L_1, L_2\}$ is a clopen partition of L , then (L, L_0) is a transitive point for INC and $(L_0 \cup L_1, L_0 \cup L_2)$ is a transitive point for INT . If $e \in L$ then (e, L) is a transitive point for EPS .*

□

Corollary 3.13 *(a) There exists a transitive point C for f_* which contains a single accumulation point $c^* \in C$, and so $C \setminus \{c^*\}$ is a set of isolated points in X . In particular, C is countable. Such a set C is never 2-decomposable. However, it can be chosen so that if c_1, \dots, c_n are distinct points of C then (c_1, \dots, c_n) is a transitive point for $f^{(n)}$.*

(b) If A is any infinite, compact, totally disconnected metric space there exists a transitive point C for f_ which is homeomorphic to A .*

Proof: (a) Any set of isolated points in a separable metric space is countable. A set with a single accumulation point is not 2-decomposable by Corollary 3.9.

Let L be a Kronecker subset and let $\{c_k : k = 1, 2, \dots\}$ be a sequence of distinct points in L converging to a point c^* in L . Let C be the points of this sequence together with the limit point. Since the sequence is convergent, the points of $C \setminus c^*$ are isolated. As it is an infinite subset of L it is a transitive point for f_* by Theorem 3.11 (b).

(b) The product $A \times L$ is perfect as well as totally, disconnected and compact metric. Hence, $A \times L$ is homeomorphic to L . Hence, L contains a homeomorphic copy of A . Such a set is an infinite closed subset of L and so is a transitive point by Theorem 3.11 (b) again.

□

A subset of X is called *strongly scrambled* when every pair of points in it is proximal and recurrent. Any Kronecker set is strongly scrambled. The map f is called *uniformly rigid* when some sequence of iterates f^{j_i} converges uniformly to the identity on X . In that case, every pair of points is recurrent and so no pair of distinct points is asymptotic. Glasner and Maon [6] have constructed uniformly rigid, weak mixing, minimal homeomorphisms on the torus. If f is uniformly rigid then every transitive point C for f is strongly scrambled.

We note that Hernandez, King and Mendez-Lango [8] have constructed Cantor sets with dense orbit in $(2^X, f_*)$.

4 Examples

We first look into an example of a Kronecker set.

Let $X = \{0, 1\}^{\mathbb{N}}$ along with the shift map σ be the one-sided shift space. Consider the induced system $(2^X, \sigma_*)$.

Let $\mathcal{L}(X)$ denote the language of X , and we define W_n to be the set of all words of length n in $\mathcal{L}(X)$. So we have,

$$W_1 = \{w_1^{(1)}, w_2^{(1)}\}, W_2 = \{w_1^{(2)}, w_2^{(2)}, w_3^{(2)}, w_4^{(2)}\}, \dots, W_n = \{w_1^{(n)}, \dots, w_{2^n}^{(n)}\}$$

where $w_i^{(j)}$ can be considered lexicographically in each W_j , i.e. in each W_j , $w_1^{(j)} < w_2^{(j)} < \dots < w_{2^{j-1}}^{(j)} < w_{2^j}^{(j)}$.

Also, we consider S_n - the group of permutations on n elements.

And let $\mathcal{U}^{n,k} \subset \mathcal{P}(W_n)$ for $1 < k \leq 2^n$, be the collection of all sets in the power set of W_n with k elements. We note that there will be ${}_2^n C_k$ of them, which we can enumerate as $U_1^{n,k}, U_2^{n,k}, \dots, U_{{}_2^n C_k}^{n,k}$. We note that $|U_j^{n,k}| = k$, and for each $u_j^{n,k} \in U_j^{n,k}$, we have $|u_j^{n,k}| = n$

Define $C \in 2^X$ as

$$C = \{w_{\rho_{11}(1)}^{(1)} w_{\rho_{11}(2)}^{(1)} w_{\rho_{12}(1)}^{(1)} w_{\rho_{12}(2)}^{(1)} w_1^{(2)} w_2^{(2)} \dots w_4^{(2)} u_1^{2,2} u_2^{2,2} \dots u_6^{2,2} u_1^{2,3} u_2^{2,3} u_3^{2,3} u_4^{2,3} u_1^{2,4} w_{\rho_{31}(1)}^{(3)} w_{\rho_{31}(2)}^{(3)} \dots w_{\rho_{31}(2^3)}^{(3)} \dots w_{\rho_{323}(1)}^{(3)} w_{\rho_{323}(2)}^{(3)} \dots w_{\rho_{323}(2^3)}^{(3)} w_1^{(4)} w_2^{(4)} w_3^{(4)} w_4^{(4)} \dots w_{2^4}^{(4)} u_1^{4,2} \dots u_{2^4 C_2}^{4,2} \dots u_1^{n,2^n} w_{\rho_{51}(1)}^{(5)} w_{\rho_{51}(2)}^{(5)} \dots w_{\rho_{51}(2^5)}^{(5)} \dots w_{\rho_{525}(1)}^{(5)} w_{\rho_{525}(2)}^{(5)} \dots w_{\rho_{525}(2^5)}^{(5)} w_1^{(6)} \dots u_1^{2n,2} \dots u_{2^{2n} C_2}^{2n,2} \dots u_{2^{2n} C_{2^{2n}-1}}^{2n,2^{2n}-1} u_1^{2n,2^{2n}} w_{\rho_{2n+11}(1)}^{(2n+1)} \dots w_{\rho_{2n+11}(2^{2n+1})}^{(2n+1)} \dots / u_j^{n,k} \in U_j^{n,k}, \rho_{k_j} \in S_{2^k} \text{ for each } n, k, j \in \mathbb{N}\},$$

i.e. C consists of all sequences with consecutive blocks of all length 1 words arranged according to all permutations in S_2 , followed by consecutive blocks of all length 2 words arranged lexicographically, followed by words of length 2 from each set containing 2, 3, 4 of these words of length 2, followed by consecutive blocks of all length 3 words arranged according to all permutations in S_8 , ..., followed by consecutive blocks of all length $2n$ words arranged lexicographically, followed by words of length $2n$ from each set containing 2, 3, ..., 2^{2n} of these words of length $2n$, followed by consecutive blocks of all length $2n+1$ words arranged according to all permutations in $S_{2^{2n+1}}, \dots$

We claim that C is compact. It is enough to see that C is closed.

Let $x = x_1 \dots x_p \dots \notin C$. This means that x cannot be realized as a sequence with a consecutive arrangement of blocks of all length n words for $n \in \mathbb{N}$ as done in C . Let t be the smallest index where this difference can be realized, and let j be largest even integer such that

$$r = t - (2 \cdot 1 + 2^2 \cdot 2 + 2 \cdot \sum_{k=2}^4 {}_4 C_k + 2^3 \cdot 3 + \dots + 2^j \cdot j + j \cdot \sum_{k=2}^{2^j} {}_{2^j} C_k) > 0,$$

and $t - r < 2^{j+1} \cdot j$.

Then, $x \in [x_1 \dots x_h]$, but $C \cap [x_1 \dots x_h] = \phi$, when $h = 2 \cdot 1 + 2^2 \cdot 2 + 2 \cdot$

$$\sum_{k=2}^4 {}_4C_k + 2^3 \cdot 3 + \cdots + 2^j \cdot j + j \cdot \sum_{k=2}^{2^j} {}_{2^j}C_k + 2^{j+1} \cdot (j+1).$$

We see that C is transitive in $(2^X, \sigma_*)$.

Take any basic open set $\langle A_1, \dots, A_q \rangle$ in the Vietoris topology on 2^X . We can always have a $p \in 2\mathbb{N}$ large enough so that the cylinders $[a_1^i \dots a_p^i] \subset A_i$, $1 \leq i \leq q$. For some $j \in 1, \dots, 2^p$, we have $U_j^{p,q} = \{a_1^1 \dots a_p^1, \dots, a_1^q \dots a_p^q\}$. Then we have $\sigma_*^h(x) \subset \langle [a_1^1 \dots a_p^1], \dots, [a_1^q \dots a_p^q] \rangle \subset \langle A_1, \dots, A_q \rangle$, where $h = 2 \cdot 1 + 2^2 \cdot 2 + 2 \cdot \sum_{k=2}^4 {}_4C_k + 2^3 \cdot 3 + \cdots + 2^p \cdot p + p \cdot \sum_{k=2}^{j-1} {}_{2^p}C_k$.

We note that each $c \in C$ is a transitive point for (X, σ) , and for every $n \in \mathbb{N}$, $C = C_1 \cup \dots \cup C_n$ such that (C_1, \dots, C_n) is a $(\sigma_*)^{(n)}$ transitive point and so C is n -decomposable.

□

Now we specialize to $X = \{0, 1\}^{\mathbb{Z}}$ with σ the shift homeomorphism.

Theorem 4.1 *There exists a transitive point C for σ_* which is a countable set with a single accumulation point $c^* \in C$. In addition, any pair of points $c_1, c_2 \in C$ is asymptotic. In particular, (c_1, c_2) is never a transitive point for $\sigma \times \sigma$.*

Proof: Call a set of finite words an EL set (equal length) if they all have the same length. Let $\{A_1, A_2, \dots\}$ be a sequence which counts all those EL sets on the alphabet $\{0, 1\}$ whose common length is odd. Let $2\ell_k + 1$ be the common length of the words in A_k . In each A_k choose a particular word a_k^* which we call the *special words*. For example, using lexicographic ordering we could choose the first word in the set A_k . Let $N_k = \sum_{j=1}^k (2\ell_j + 1)$.

We now construct inductively a sequence of EL sets $\{B_1, \dots\}$ for which N_k is the common length of the words in B_k and there is a distinguished element $b_k^* \in B_k$. Each member of B_k is a concatenation of blocks of length

$2\ell_1 + 1, \dots, 2\ell_k + 1$. We refer to these as the first block, the second, up to the k^{th} block.

Let $B_1 = A_1$ and $b_1^* = a_1^*$.

Inductively, we concatenate, defining B_{k+1} to be the union of the two sets $\{b_k^*\} \cdot A_{k+1}$ and $B_k \cdot \{a_{k+1}^*\}$. The intersection of these two sets is the single word $b_k^* \cdot a_{k+1}^*$ which we define to be b_{k+1}^* .

Thus, $b_{k+1}^* = a_1^* \cdot a_2^* \dots a_k^* \cdot a_{k+1}^*$.

Inductively it is clear that for each word in B_{k+1} at most one block is not a special word.

Define $c \in C$ when $c_i = 0$ for all $i \leq 0$ and when $c_{[1, \dots, N_k]} \in B_k$ for all k . Let c^* be the special element of C with $c_i^* = 0$ for all $i \leq 0$ and $c_{[1, \dots, N_k]}^* = b_k^*$.

If $a_k \in A_k \setminus \{a_k^*\}$ and the k^{th} block of $c \in C$ is a_k then every other block on (the positive side) is a special word. Thus, c agrees with c^* except in finitely many places and it is clear that c is an isolated point of C . Thus, the points of any finite subset of C all agree with c^* after some finite coordinate level. It follows that any two points are asymptotic. This in turn implies that no pair in C defines a transitive point for $\sigma \times \sigma$.

Finally, we observe that the words from the $-\ell_{k+1}$ to the $+\ell_{k+1}$ coordinate occurring in the set $\sigma^{N_k + \ell_{k+1} + 1}(C)$ are exactly the words of A_{k+1} centered in the middle. Since the sequence $\{A_k\}$ lists all EL sets of words of odd length it follows that the iterates $\sigma_*^j(C)$ approach every element of 2^X arbitrarily closely. That is, C is a transitive point for σ_* .

□

We can obtain Cantor set examples by using the *asymptotic extension technique*. For $B \in 2^X$ with $X = \{0, 1\}^{\mathbb{Z}}$ let $B_N = \pi_N^{-1}(\pi_N(B))$ where $\pi_N : \{0, 1\}^{\mathbb{Z}} \rightarrow \{0, 1\}^{[N, \infty)}$ is the coordinate projection. It is clear that B and B_N are asymptotic points of 2^X , with respect to σ_* , since they agree beyond the N^{th} coordinate. Furthermore, every point of B_N is asymptotic to a point of B . In particular, if B is a transitive point for σ_* then each B_N is as well.

Corollary 4.2 *There exists a transitive point C for σ_* which is a Cantor set such that any pair of points $c_1, c_2 \in C$ is asymptotic. In particular, (c_1, c_2) is never a transitive point for $\sigma \times \sigma$.*

Proof: Let B be the countable set constructed for Theorem 4.1 so that every point of B is asymptotic to c^* . Clearly, every point of $C = B_N$ is

asymptotic to c^* . Since the coordinates below N are arbitrary B_N is a Cantor set.

□

Remark: By extending the idea of Theorem 4.1 one can construct B to be a disjoint union of sets B_1, \dots, B_n each an infinite countable set contained in a single asymptotic class and so that (B_1, \dots, B_n) is a transitive point for $(\sigma)^{(n)}$. Then B_N is a Cantor set n -decomposable transitive point which meets only n asymptotic classes and so is not $n + 1$ -decomposable.

Using this asymptotic extension idea in a different context we can obtain examples of transitive points in 2^X with nontrivial components.

We specialize to $X = \mathbb{R}^2/\mathbb{Z}^2$ with $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ the canonical projection. We define the metric d on $\mathbb{R}^2/\mathbb{Z}^2$ by $d(a + \mathbb{Z}^2, b + \mathbb{Z}^2) = \min\{|a - b + z| : z \in \mathbb{Z}^2\}$. Hence, π has Lipschitz constant 1. Any closed $C \subset \mathbb{R}^2/\mathbb{Z}^2$ of diameter less than 1 is contained in a ball in $\mathbb{R}^2/\mathbb{Z}^2$ and by lifting the ball we obtain a closed $C^+ \subset \mathbb{R}^2$ such that $\pi : C^+ \rightarrow C$ is a homeomorphism. If $C \subset \mathbb{R}^2/\mathbb{Z}^2$ is a Cantor set of any diameter, we can choose a clopen partition of C by sets of diameter less than one and lift each separately to obtain a Cantor set $C^+ \subset \mathbb{R}^2$ such that $\pi : C^+ \rightarrow C$ is a homeomorphism.

Now let t be a Thom torus map on $\mathbb{R}^2/\mathbb{Z}^2$, e.g. the diffeomorphism induced by the linear map T with matrix $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$. This is an Anosov diffeomorphism and so is a factor of the shift map. In particular, it is weak mixing. The eigenvalues $\lambda_{\pm} = (5 \pm \sqrt{21})/2$ satisfy $4 < \lambda_+ < 5$ and $0 < \lambda_- < \frac{1}{2}$. Let v_+ and v_- be corresponding eigenvectors of unit length. The lines parallel to v_- and to v_+ project to the stable and unstable foliations. In particular, for any $a \in \mathbb{R}^2$ and any real number s

$$|T^j(a) - T^j(a + sv_-)| = |s|(\lambda_-)^j < |s|2^{-j}. \quad (4.1)$$

In particular, all points on the stable manifold of $\pi(a)$, i.e. $\pi(a + \mathbb{R}v_-)$, are asymptotic with respect to t .

Let L be a Kronecker subset of $\mathbb{R}^2/\mathbb{Z}^2$ for t . There exists a sequence $j_k \rightarrow \infty$ such that $t^{j_k}|L$ converges to the inclusion map of L into $\mathbb{R}^2/\mathbb{Z}^2$. Hence, no two distinct points of L are asymptotic. Hence, the stable manifolds of any two distinct points in L are disjoint. Let L^+ be a Cantor set in \mathbb{R}^2 such that $\pi|L^+ \rightarrow L$ is a homeomorphism. Define the map $Q : L^+ \times \mathbb{R} \rightarrow \mathbb{R}^2/\mathbb{Z}^2$ by $Q(a, s) = \pi(a + sv_-)$. The map Q is clearly continuous. It is injective

because the v_- lines through distinct points of L^+ do not intersect. For $M > 0$ let $C_M = Q(L^+ \times [-M, M])$. By compactness, the restriction of Q is a homeomorphism from $L^+ \times [-M, M]$ onto C_M . From equation (??) it follows that the Hausdorff distance between $T^j(L^+)$ and $T^j(L^+ + [-M, M]v_-)$ converges to zero. Hence, the distance between $t_*^j(L)$ and $t_*^j(C_M)$ converges to zero. That is L and C_M are asymptotic points of 2^X . Since L is a transitive point, C_M is as well. Thus, we have proved:

Theorem 4.3 *There exists a transitive point C for t_* which is homeomorphic to $L \times [0, 1]$ where L is a Cantor set. In particular, every component of C is an interval.*

□

Remark: Note that $\{C_M : M = 1, 2, \dots\}$ is an increasing sequence of transitive points for t_* whose union is dense in $\mathbb{R}^2/\mathbb{Z}^2$.

In examples of this sort any pair of points which lie in the same component of the transitive point C are asymptotic. The question arises whether this is always true. We obtain a partial result suggesting that this is true.

For a compact metric space X define the *component diameter* to be

$$Cdiam(X) = \sup\{d(x, y) : x \text{ and } y \text{ lie in the same component of } X\}. \quad (4.2)$$

Equivalently, $Cdiam(X)$ is the supremum of the diameters of the components of X .

Lemma 4.4 *If $\{B_i\}$ is a sequence in 2^X converging to a totally disconnected $C \in 2^X$ then*

$$\lim_{i \rightarrow \infty} Cdiam(B_i) = 0. \quad (4.3)$$

Proof: Given $\epsilon > 0$ choose a clopen partition of C by elements of diameter less than ϵ and enlarge each to an open subset of X . That is, we can obtain a disjoint open sets U_1, \dots, U_n whose union U contains C . There exists N so that $i \geq N$ implies $B_i \subset U$ and so each component is entirely contained in one of the U_i 's. Hence, $i \geq N$ implies that $Cdiam(B_i) < \epsilon$.

□

We conjecture that if C is a transitive point for f_* on 2^X , then

$$\lim_{i \rightarrow \infty} \text{Cdiam}((f_*)^i(C)) = 0.$$

The closest we can get is the following result

Theorem 4.5 *Assume f is a homeomorphism. C is a transitive point for f_* on 2^X and $B \in 2^X$ then there exists a sequence $\{j_i\}$ of integers tending to infinity such that:*

$$\begin{aligned} \lim_{i \rightarrow \infty} \{(f_*)^{j_i}(C)\} &= B, \quad \text{and} \\ \lim_{i \rightarrow \infty} \text{Cdiam}((f_*)^{j_i}(C)) &= 0. \end{aligned} \tag{4.4}$$

Proof: Let L be a Cantor set transitive point for f_* and choose sequences n_p, m_q so that $(f_*)^{n_p}(C) \rightarrow L$ and $(f_*)^{m_q}(L) \rightarrow B$ as $p, q \rightarrow \infty$. Diagonalize. That is, given $i > 0$ choose q_i so that $d(B, (f_*)^{m_{q_i}}(L)) < 1/2i$ and then p_i so that $d((f_*)^{m_{q_i}}(L), (f_*)^{m_{q_i}+n_{p_i}}(C)) < 1/2i$ and $\text{Cdiam}((f_*)^{m_{q_i}+n_{p_i}}(C)) < 1/i$. Because f is a homeomorphism $(f_*)^{m_{q_i}}(L)$ is a Cantor set and so Lemma 4.4 implies that p_i exists. Let $j_i = m_{q_i} + n_{p_i}$. \square

If $H(K)$ is the homeomorphism group of the Cantor set K then $H(K)$ acts on $\mathcal{C}(K, X)$ by $tg = g \circ t^{-1}$. This action commutes with f_* . The map $\text{Im} : \mathcal{C}(K, X) \rightarrow 2^X$ is constant on the $H(K)$ orbits. Let $\mathcal{C}_{in}(K, X)$ be the set of injective continuous maps. This is a dense G_δ subset of $\mathcal{C}(K, X)$ and it is invariant under f_* . It maps to $\text{CANTOR} \subset 2^X$ the dense G_δ subset of Cantor sets in X . Furthermore the fibers of the map $\mathcal{C}_{in}(K, X) \rightarrow \text{CANTOR}$ are exactly the $H(K)$ orbits.

5 Minimality and Distality for the Induced Dynamics of Compact Systems

The system (X, f) is *minimal* if every $x \in X$ is a transitive point. We note that X is always a fixed point for $(2^X, f_*)$, and so the system $(2^X, f_*)$ can never be minimal. Here we talk of minimality in a weaker sense. In this section X will always be a compact metric space.

Definition 5.1 *The system (X, f) is said to be backward minimal if the negative semiorbit of x , $O^-(x) = \{y \in X : f^n(y) = x \text{ for some } n \in \mathbb{N}\}$, is dense in X for every $x \in X$.*

Equivalently, (X, f) is backward minimal if for every opene U , $X = \bigcup_{n=1}^{\infty} f^n(U)$.

Remark: We can immediately make the following conclusions:

1. A minimal system is always backward minimal and if f is a homeomorphism, then minimal and backward minimal are equivalent.
2. A backward minimal system is always topologically transitive.
3. A system (X, f) is called *exact* if for every opene U in X , there exists an $n \in \mathbb{N}$ such that $f^n(U) = X$. An exact system is always backward minimal.

A simple example of a backward minimal system is the one-sided shift space on finite symbols. It is easy to see that this shift space will be exact but not minimal. Also, the two-sided shift space on finite symbols, where the shift map is now a homeomorphism, fails to be backward minimal. The irrational rotation is a minimal system so is backward minimal but is not exact.

Theorem 5.2 *A backward minimal system (X, f) need not be weakly mixing. In particular, in this case $(2^X, f_*)$ need not be topologically transitive.*

Proof: Let X be the closure in $\{0, 1\}^{\mathbb{N}}$ of the sequences of the form

$$0^k 10^{3^{n_1}} 10^{3^{n_2}} 10^{3^{n_3}} 1 \dots 10^{3^{n_i}} 1 \dots$$

where $k, n_i \in \mathbb{N}$. Consider the shift map σ on X . We see that for any cylinder, a finite image will be a cylinder of some sequences that start with 1, and consequently the images of this cylinder will cover the whole of X . So this system (X, σ) is backward minimal.

However, there is no $n \in \mathbb{N}$ for which $\sigma^n[001] \cap [1] \neq \emptyset$ and also $\sigma^n[010] \cap [1] \neq \emptyset$ simultaneously. So our system (X, σ) is not weakly mixing. By Theorem 2.4, $(2^X, \sigma_*)$ cannot be topologically transitive.

□

Theorem 5.3 *For a dynamical system (X, f) the following are equivalent:*

- (i) (X, f) is exact.

(ii) $(2^X, f_*)$ is exact.

(iii) $(2^X, f_*)$ is backward minimal.

(iv) The negative orbit $O^-(\{X\})$ of the point X is dense in 2^X .

(v) $i_X(X) \subset 2^X$ is contained in the closure of $\bigcup_{i \geq 0} (f_*)^{-i}(\{X\})$ in 2^X .

Proof: It is obvious that (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v).

(v) \Rightarrow (i): We are assuming that the backward orbit of the point $X \in 2^X$ meets every neighborhood of $\{x\} \in 2^X$ for any $x \in X$. Thus, any opene subset of X contains a closed set A which maps onto X under some iterate of f . That is, f is exact.

(i) \Rightarrow (ii): Since f is exact, it is surjective and so $f^n(U) = X$ implies $f^i(U) = X$ for $i \geq n$. Now, given $A \in 2^X$, take a Vietoris neighbourhood of A i.e. a list of closed sets C_1, \dots, C_k covering A and such that each interior meets A . There exists $n \in \mathbb{N}$ so that $f^n(\text{Int}C_i) = X$ for $i = 1, \dots, k$. For any $B \in 2^X$, let $D_i = (f^{-n}|C_i)(B)$. Each $D_i \subset C_i$ and meets $\text{Int}C_i$. Hence, $A_1 = \bigcup_{i=1}^k D_i$ is in the Vietoris neighbourhood of A , while $f^n(A_1) = B$. Hence, f_* is exact.

□

We recall the *proximal relation* $P \subset X \times X$ in (X, f) . We say that $(x, y) \in P$ if $\liminf d(f^n x, f^n y) = 0$, $n \in \mathbb{N}$. Equivalently, for any $\epsilon > 0$ there is a $n \in \mathbb{N}$ such that $d(f^n x, f^n y) < \epsilon$. The relation P is reflexive, symmetric, and f invariant. The system (X, f) is said to be *distal* if $P = \Delta$, i.e. there are no non-trivial proximal pairs. It is obvious that in such a case our map will be a homeomorphism.

The system (X, f) is *equicontinuous* if the maps of X defined by the iterates of f form an equicontinuous family. That is, for every $\epsilon > 0$, there is a $\delta > 0$ such that whenever $d(x, x') < \delta$ then $d(f^n x, f^n x') < \epsilon$ for all $n \in \mathbb{N}$.

We know that equicontinuous systems are distal, and it is well known that the converse fails. We note that the system (X, f) is equicontinuous if and only if the system $(2^X, f_*)$ is equicontinuous [17]. However, we observe that (X, f) distal need not imply that $(2^X, f_*)$ is distal.

For the unit disc D , consider the map $f : D \rightarrow D$ defined by $f(r, \theta) = (r, r + \theta)$. Clearly, (D, f) is distal. Let $r_n \mapsto i$, where r_n are rationals and i

an irrational, and consider $A = \{(r_n, 0) : n \in \mathbb{N}\} \cup \{(i, 0)\} \in 2^X$. The orbit closure of A will consist of the set B with the full circle at the irrational radius i and periodic orbits at each rational radius r_n . So A is not in the orbit closure of B .

A point is called *almost periodic* if its orbit closure is a minimal system. It is known that a distal system can be realized as a union of minimal systems, and so every point is an *almost periodic point*. Since a distal system is pointwise almost periodic, and the point A fails to be almost periodic, $(2^D, f_*)$ cannot be distal.

To this end, we recall the definition of $RP \subset X \times X$ the *regionally proximal relation*. We say that $(x, y) \in RP$ if there are sequence $\{x_n\}$ and $\{y_n\}$ with $x_n \rightarrow x$ and $y_n \rightarrow y$ and integers k_n such that $d(f^{k_n}(x_n), f^{k_n}(y_n)) \rightarrow 0$. It is known that (X, f) is equicontinuous if and only if $RP = \Delta$. That is, whenever $x \neq y$, $(x, y) \notin RP$.

A classical theorem of Ellis says that (X, f) is distal if and only if the product flow $(X \times X, f \times f)$ is pointwise almost periodic. That is, every orbit closure in $X \times X$ is minimal.

We now prove when $(2^X, f_*)$ will be distal.

Theorem 5.4 *The following are equivalent:*

- (i) (X, f) is equicontinuous.
- (ii) $(2^X, f)$ is equicontinuous
- (iii) $(2^X, f)$ is distal.

Proof: It is easy to show that (i) \implies (ii), and as since equicontinuous systems are distal, (ii) \implies (iii).

For (iii) \implies (i), suppose that (X, f) is not equicontinuous. Then there are x and y in X with $x \neq y$ and $(x, y) \in RP$. Then we have $x_n \rightarrow x$, $y_n \rightarrow y$ and $k_n \in \mathbb{Z}$ with $(f^{k_n}(x_n), f^{k_n}(y_n)) \rightarrow (z, z)$ for some $z \in X$. Let $C = \{x_1, x_2, \dots, x\}$ and $D = \{y_1, y_2, \dots, y\}$. We may suppose $C \cap D = \emptyset$. We show that the orbit closure of (C, D) in $2^X \times 2^X$ is not minimal. Let (a subsequence of) $(f_* \times f_*)^{k_n}(C, D) \rightarrow (C', D')$. Then $C' \cap D' \neq \emptyset$ (in the terminology of Proposition 1.3, $(C', D') \in INT$). Now if the orbit closure of (C, D) were minimal, there would be a sequence $\{l_m\}$ with $(f_* \times f_*)^{l_m}(C', D') \rightarrow (C, D)$. But in this case $C \cap D \neq \emptyset$, a contradiction.

□

A system (X, f) is *sensitive* if there exists a $\delta > 0$ such that for any $x_1 \in X$ and opene set $U \ni x_1$, there exists $x_2 (\neq x_1) \in U$ and $m \in \mathbb{N}$ such that $d(f^m(x_1), f^m(x_2)) > \delta$.

It is known that minimal systems are either equicontinuous or sensitive. Various forms of sensitivities for the induced system $(2^X, f_*)$ are studied by Sharma and Nagar in [16], and among other things there is an example of a sensitive (X, f) for which $(2^X, f_*)$ is not sensitive.

6 (X, f) Can Be Isomorphic to Its Own Induced System

In this section we will again restrict attention to compact systems (X, f) .

The dynamical system $(2^X, f_*)$ induces the system $(2^{2^X}, f_{**})$, and in many cases we believe that this will be a transfinite process. Hence, it would be nice to characterize those compact dynamical systems (Y, g) which are of the form $(2^X, f_*)$ for some (X, f) , (necessarily a compact system since X is isometric with a closed subset of 2^X). In this section, we look into some cases where this process of inducing definitely gives us nonconjugate dynamics or where this process of inducing stabilizes.

A subset $A \subset X$ is called *+ invariant* when $f(A) \subset A$ and invariant when $f(A) = A$. Observe that if A is closed then it is invariant if and only if as an element of 2^X it is a fixed point for f_* . Hence, if f is transitive then if A is a fixed point for f_* then either $A = X$ or A has empty interior. If f is *totally transitive*, i.e. every power f^n for $n = 1, \dots$ is transitive, then if A is a periodic point for f_* , and hence a fixed point for some f_*^n , then either $A = X$ or A has empty interior. In particular, a weak mixing system is totally transitive and so this then holds.

If $F \subset 2^X$ is a closed invariant set for f_* then as an element of 2^{2^X} it is a fixed point and hence $\vee(F) = \bigcup F$ is a fixed point for f_* , i.e. an invariant set for f . In particular the union of any periodic orbit of f_* is a fixed point of f_* . Notice that the union of any finite collection of periodic points of f_* is a periodic point (with period dividing the l.c.m. of the periods of the members of the collection). The intersection, if non-empty, is also a periodic point.

Recall that if the periodic points are dense in X then the same holds for 2^X . The converse is not true as we will see below. Any odometer (X, f) is minimal, and equicontinuous and (so is not totally transitive). Any point of x can be approximated by a periodic element of 2^X with nonempty interior and so with the union of the orbit equal to X . It follows that the periodic points are dense in 2^X .

Theorem 6.1 *Let (X, f) be a compact dynamical system. The periodic points are dense in $(2^X, f_*)$ if their closure contains $i(X) \subset 2^X$.*

If f is totally transitive and the periodic points are dense in 2^X then there are infinitely many distinct fixed points for f_ .*

Proof: If $\{x_1, \dots, x_k\} \subset X$ and $\epsilon > 0$ then there exist periodic points $\{A_1, \dots, A_k\} \subset 2^X$ such that $d(A_i, \{x_i\}) < \epsilon$ for $i = 1, \dots, k$. Then $A = A_1 \cup \dots \cup A_k$ is a periodic point with $d(A, \{x_1, \dots, x_k\}) < \epsilon$. Hence, the closure of the periodic points contains $FIN(X)$ which is dense in 2^X .

The unions of each of a finite collection of periodic orbits for f_* , other than the fixed point X itself, are closed, nowhere dense sets since f is totally transitive. Each of the unions is a fixed point. There is an open set $U \subset X$ disjoint from all of them and it contains a periodic point A for f_* . The union of the associated periodic orbit is not contained in the previous union and so is distinct from the previous fixed points.

□

Corollary 6.2 *If (X, f) is a mixing subshift of finite type then it is not isomorphic to $(2^Y, g_*)$ for any compact system (Y, g) .*

Proof: A subshift of finite type has dense periodic points and is expansive. Hence, it cannot admit infinitely many distinct fixed points.

□

An *inverse system* is a sequence between compact metrizable spaces: $\{\pi_i : X_{i+1} \rightarrow X_i : i = 1, 2, \dots\}$. The associated *inverse limit* is the compact metrizable space $X_\infty = \{x \in \prod_i X_i : x_i = \pi_i(x_{i+1}) \text{ for } i = 1, 2, \dots\}$, which is metrizable when the X_i 's are. The projection map the restriction map to the i^{th} coordinate is denoted $\pi^i : X_\infty \rightarrow X_i$. If A is a closed subset of X_∞ we let $F_i = \pi^i(A)$. If $x \in X_\infty$ is such that $\pi_i(x) \in F_i$ for all i then $\{F \cap (\pi_i)^{-1}(x_i)\}$ is a decreasing sequence of nonempty compacta and so the intersection, which is $F \cap \{x\}$ is nonempty. That is, $x \in F$. Thus, the restrictions $\pi^i : F \rightarrow F_i$ and $\pi_i : F_{i+1} \rightarrow F_i$ are all surjective and so F is the inverse limit of the system $\{\pi_i : F_{i+1} \rightarrow F_i\}$. It thus follows that the map from $\{(\pi_i)_* : 2^{X_{i+1}} \rightarrow 2^{X_i}\}$ is an inverse system. Furthermore the maps $(\pi^i)_* : 2^{X_\infty} \rightarrow 2^{X_i}$ induce a natural homeomorphism between 2^{X_∞} and the inverse limit of the system $\{(\pi_i)_* : 2^{X_{i+1}} \rightarrow 2^{X_i}\}$. Refer to [9, 15] for more details.

Now let X be any compact metric space. Define $X_1 = 2^X$, $X_2 = 2^{2^X} = 2^{X_1}$ and $\pi_1 = \vee : X_2 \rightarrow X_1$. Inductively, define $X_{i+1} = 2^{X_i}$ and $\pi_i = (\pi_{i-1})_* : X_{i+1} = 2^{X_i} \rightarrow 2^{X_{i-1}} = X_i$. Let X_∞ denote the inverse limit of this inverse system. As remarked in the previous paragraph, there is a natural homeomorphism from 2^{X_∞} to the inverse limit of the system $\{(\pi_i)_* : 2^{X_{i+1}} \rightarrow 2^{X_i}\}$ which is our original system with the index shifted by one. That is, there is a natural homeomorphism $q : 2^{X_\infty} \rightarrow X_\infty$.

If (X, f) is a compact dynamical system then letting $f_1 = f_*$ and $f_{i+1} = (f_i)_*$ we obtain an inverse limit of dynamical systems with f_∞ the map induced on X_∞ and q is an isomorphism from $(2^{X_\infty}, (f_\infty)_*)$ to (X_∞, f_∞) . Since the inverse limit of weak mixing systems is weak mixing, (X_∞, f_∞) is weak mixing when (X, f) is. Furthermore, the pair of action maps $(\pi_1)_* : 2^{X_\infty} \rightarrow 2^{2^X}$ and $\pi_1 : X_\infty \rightarrow 2^X$ maps the isomorphism q to \vee . Thus, we have proved the following result.

Theorem 6.3 *If (X, f) is a compact dynamical system then there $(2^X, f_*)$ is a factor of a compact dynamical system (X_∞, f_∞) which is isomorphic to $(2^{X_\infty}, (f_\infty)_*)$. Furthermore, if (X, f) is weak mixing then (X_∞, f_∞) is weak mixing.*

□

If $\{n_i\}$ is a strictly increasing sequence of positive integers with $n_i | n_{i+1}$ then the inclusions of $n_{i+1}\mathbb{Z} \rightarrow n_i\mathbb{Z}$ induces an inverse system $\{\pi_i : \mathbb{Z}/n_{i+1}\mathbb{Z} \rightarrow \mathbb{Z}/n_i\mathbb{Z}\}$ of epimorphisms of finite rings. The inverse limit $\mathbb{Z}_{\{n_i\}}$ is a compact ring which is topologically a Cantor set. The associated *odometer* or *adding machine* uses the translation map $t(x) = x + 1$. Every clopen subset is a periodic element for $(\mathbb{Z}_{\{n_i\}}, t_*)$, as it is the preimage of a finite subset of $\mathbb{Z}/\mathbb{Z}_{\{n_i\}}$ for sufficiently large i . Hence, from Theorem 6.1 it follows that the periodic points are dense in $\mathbb{Z}_{\{n_i\}}$. On the other hand, $(\mathbb{Z}_{\{n_i\}}, t)$ is minimal and so has no periodic points.

An interesting question is which dynamical systems can be represented as $(2^X, f_*)$ for some system (X, f) . This is far from settled, but we have obtained some obstructions to such. Of course, 2^X can never be minimal. Theorem 5.3 tells us that it can't be backwards minimal unless it is exact, and by Theorem 5.4, it can't be distal unless it is equicontinuous.

Moreover, $(2^X, f_*)$ has some structure which apparently does not occur in most dynamical systems, namely the relations *INT*, *INC*, and *EPS* defined

in Proposition 1.3. These are closed $f_* \times f_*$ invariant relations, and an interesting problem is to determine their dynamical significance.

Acknowledgement: This work was done when the third author visited University of Maryland. She acknowledges the hospitality of the Mathematics Department at the University.

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